AIAA 80-1598R

# Analysis of Minimum Noise Landing Approach Trajectory

H. Ohta*
*Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Japan*

**Optimal control theory is used to determine thrust and glide path angle programs for minimum time and minimum noise landing approaches of aircraft modeled by approximated point-mass dynamics. The necessary conditions for optimum programs are derived, and numerical results are presented for a jet fighter. Significant noise abatement and savings in time are obtained compared with the reference thrust and glide path angle programs. The minimum time solution indicates: 1) the thrust is a dominant control, and glide path angle $\gamma$ has little contribution to the performance index; and 2) the optimum path consists mainly of a descending segment with a constant velocity, which is a singular solution, and a decelerating segment with $\gamma_{max}$. The optimum thrust switches from its maximum to its minimum on the latter segment. The minimum noise solution indicates: 1) the optimum thrust achieves neither its maximum nor its minimum, but varies continuously so as to yield a smooth EPN (effective maximum perceived noise) level curve; and 2) the optimum path consists of a decelerating segment with $\gamma_{max}$ and a descending one with $\gamma_{min}$. Comparisons with a two-segment approach are given.**

## Nomenclature

| | |
|---|---|
| $C_1, C_2$ | = coefficients of $V^2$ and $1/V^2$ in $D$ |
| $C_{D0}$ | = drag coefficient for zero lift |
| $C_N, C_T$ | = constants defined in Eqs. (13) and (A8), respectively |
| $D, C_D$ | = aerodynamic drag force (in kg) and drag coefficient, respectively |
| EPN | = effective maximum perceived noise |
| $f_T$ | = variable defined in Eq. (20) |
| $g$ | = acceleration of gravity = 9.81 m/s$^2$ |
| $H$ | = variational Hamiltonian |
| $h$ | = altitude, m |
| $J$ | = performance index |
| $K_i$ | = coefficients of the performance index defined in Eq. (10), $i=1,...,4$ |
| $L, C_L$ | = aerodynamic lift force (in kg) and lift coefficient, respectively |
| $m$ | = mass of vehicle, kg s$^2$/m |
| PN | = maximum perceived noise |
| $R$ | = distance from vehicle, m |
| $S$ | = aerodynamic reference area of vehicle, m$^2$; velocity constant in Eq. (24) |
| $T$ | = thrust of vehicle, kg |
| $t$ | = time, s |
| $V$ | = velocity of vehicle, m/s |
| $W$ | = weight of vehicle = $mg$, kg |
| $x$ | = horizontal distance, m |
| $\alpha$ | = angle between vectors $V$ and $T$, rad |
| $\Gamma$ | = coefficient of $\gamma$ in Hamiltonian |
| $\gamma$ | = flight path angle, rad |
| $\eta$ | = induced drag parameter |
| $\lambda_1, \lambda_2$ | = Lagrange multipliers for $V$ and $h$, respectively |
| $\mu$ | = Lagrange multiplier for velocity constraint |
| $\rho$ | = atmospheric density = 0.125 kg s$^2$/m$^4$ |
| $(\ )_0$ | = initial value |
| $(\ )_f$ | = final value |
| $(\dot{\ })$ | = d/d$t$ or d/d$x$ |

## Introduction

DURING the past few years, much effort has been directed at alleviating aircraft noise during takeoff and landing, and terminal congestion. These efforts have been directed not only at improving the performance of engines and aircraft, but also at the development of efficient terminal guidance and aircraft control systems. Aircraft performance optimization studies have been mainly problems associated with minimizing the time of climb, maximizing range, and minimizing fuel consumption in the vertical plane,[1-3] and minimizing the time of turn[4] and optimum guidance[5] problems in the horizontal plane. Schultz[3] and Vincent,[6] for example, considered the minimum-time-descent problem, but they did not include final boundary conditions at touchdown due to their formulation from a high altitude. As far as minimizing noise is concerned, takeoff is more critical than landing. The optimum takeoff program for minimum aircraft noise is simply a steep ascent with maximum thrust,[7] which is the same as that of minimizing the time of climb. There is, however, no obvious solution for minimizing the noise during landing. Jacob[8] carried out optimization studies on a STOL aircraft, using an approximate power series expansion of the control variables, but his results do not apply to the minimum noise problem because the criterion that he used is a weighted sum of the noise and deviation from a specified landing path.

The direct application of the maximum (or minimum) principle for the derivation of optimum solutions usually leads to computational difficulties except for simple problems. An alternative approach is to use numerical or expansion techniques such as the steepest ascent method,[9] or a singular perturbation method.[10,11] The use of a gradient algorithm appears to be simple and straightforward, but difficulties arise in attaining convergence. Also, the qualitative properties of the optimum solution are obscure. These difficulties may be alleviated when the optimum solution of a simplified problem is used as an initial estimate in the algorithms. The singular perturbation technique discussed by Kelley[10] requires a solution of reduced-order problems.

In this paper, a discussion is presented of the approach maneuvers of an aircraft in the vertical plane. A derivation is then given of the basic optimum solutions for minimizing the time and noise during the landing approach. The direct use of the maximum principle is made so as to obtain preliminary optimum solutions for the simplified dynamics of the aircraft.

## Aircraft Model

The point-mass equations of an aircraft in the vertical plane are given by

$$m\dot{V} = T\cos\alpha - D - W\sin\gamma$$

$$mV\dot{\gamma} = T\sin\alpha + L - W\cos\gamma$$

$$\dot{h} = V\sin\gamma \qquad \dot{x} = V\cos\gamma$$

Presented as Paper 80-1598 at the AIAA Atmospheric Flight Mechanics Conference, Danvers, Mass., Aug. 11-13, 1980; submitted Oct. 15, 1980; revision received Sept. 15, 1981. Copyright © American Institute of Aeronautics and Astronautics, Inc., 1980. All rights reserved.

*Lecturer, Department of Aeronautical Engineering. Member AIAA.

where it is assumed that $\alpha$, $\gamma \ll 1$, and that $\dot{\gamma}=0$ $(L=W)$ except for short segments of the approach path. The aircraft model to be optimized is expressed as

$$\dot{V}=(T-D)/m-g\gamma \qquad \dot{h}=V\gamma \qquad \dot{x}=V \tag{1}$$

where all the initial and final conditions of the state variables, $V$, $h$, and $x$, are specified.

$$V(t_0)=V_0 \qquad h(t_0)=h_0 \qquad x(t_0)=x_0$$

$$V(t_f)=V_f \qquad h(t_f)=h_f \qquad x(t_f)=x_f \tag{2}$$

The constraints of the control variables, $T$ and $\gamma$, are

$$T_{\min} \leq T \leq T_{\max} \qquad \gamma_{\min} \leq \gamma \leq \gamma_{\max} \tag{3}$$

The drag coefficient is assumed to obey the parabolic formula $C_D = C_{D0} + \eta C_L^2$, and the drag $D$ is then given by

$$D(V) = C_1 V^2 + C_2/V^2 \tag{4}$$

where

$$C_1 = \tfrac{1}{2}\rho S C_{D0} \qquad C_2 = 2\eta W^2/\rho S$$

are constants if the change of $\rho$ with altitude is neglected. The problem of concern is to determine the control variables $T$ and $\gamma$, which minimize a performance index $J$ subject to Eqs. (1-3).

## Reference Approach Path

The aircraft data used for the numerical analysis are those of the F-104 G airplane (using BLC) for a landing approach configuration (landing-gears and flaps down). The dimensions of the airplane are $W=7180$ kg and $S=18.2$ m$^2$, and the coefficients in Eq. (4) are $C_1=0.226$ and $C_2=5.20\times 10^6$.

A reference approach path of the type shown in Fig. 1 is used so as to yield the appropriate boundary conditions. Here it is assumed that the initial approach of the aircraft is a $-6$ deg glide path angle, which changes to $-3$ deg at the outer marker point. The overall horizontal distance is 15 km. The touchdown speed is $1.15V_{\text{stall}}$, while the initial velocity is the flap-down speed. For this reference approach path, the values of the boundary conditions in Eq. (2) are

$$V_0=124 \qquad h_0=1197 \qquad x_0=0$$

$$V_f=77.5 \qquad h_f=0 \qquad x_f=15{,}000 \tag{5}$$

The time histories of the reference velocity $V$, and thrust $T$, along the reference path are also shown in Fig. 1. The time required for landing is 153.32 s. The thrust is 2030 kg until the airplane reaches a point about 1.5 km from touchdown, where $T$ is decreased to 1500 kg so as to meet the boundary conditions. Since the time histories of the approach path, thrust, and velocity shown in this figure approximate an actual situation, the boundary values given in Eq. (5) are adequate.

## Summary of Minimum Time Landing Approach

The results of the minimum time problem will now be summarized so they may be compared with those of the minimum noise case.

The optimum solution was derived, and numerical results for the three constraint values of $\gamma$ in Eq. (3), were illustrated in Ref. 12. By comparing these optimum trajectories with the reference trajectory N0, it is seen that the landing time is reduced by about 32 s, or 20%. It should be noted, also, that these optimum trajectories have a segment generated by the singular control[13] of $\gamma$. However, the time differences between the optimum trajectories, and those generated using the bang-bang control of $\gamma$, are small. This is true even if $\gamma$ does not satisfy the necessary conditions of optimality. The dominant control is the thrust $T$, not $\gamma$. It is also seen that the greater the upper bound of $\gamma$ becomes, the smaller is the time required for landing.

In the minimum time problems, the optimum thrust must be either at its maximum or its minimum regardless of takeoff and landing. The thrust is at maximum if the final velocity is free. If it is specified, the aircraft must decelerate near the final point. Minimum thrust is needed, therefore, in order to fly with maximum thrust for as long as possible. Optimum paths for relatively long flight distances consist, therefore, of three segments of $\gamma_{\min}$, a singular solution, and $\gamma_{\max}$ having acceleration, a constant velocity, and deceleration, respectively.

It is to be noted that the initial point is not the starting point of the approach, but only one reached by decelerating from a cruising speed. This implies that the selection of the initial conditions are somewhat arbitrary, and that it is not adequate to accelerate again between the approach path. This consideration leads to prolonged singular segments, where the flight velocities are constant. It is natural, therefore, to guide an aircraft on to the extended paths, and to make it descend along the paths to the final segment with $-3$ deg of the ILS beam. If the constant velocity on a singular arc is given, other variables there (and thereafter) are uniquely determined. For this reason, a two-segment approach path with the singular angle and $-3$ deg was selected for this study.

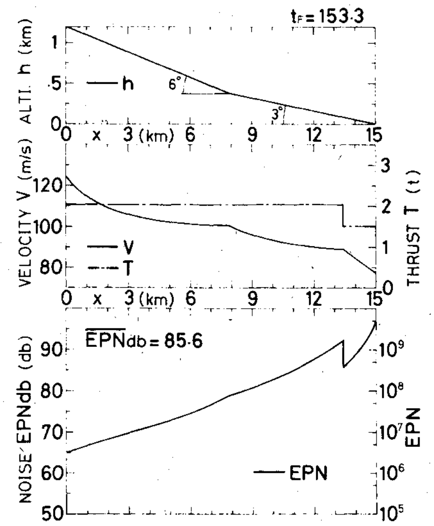


Fig. 1 Case N0 (reference noise for the two-segment path).

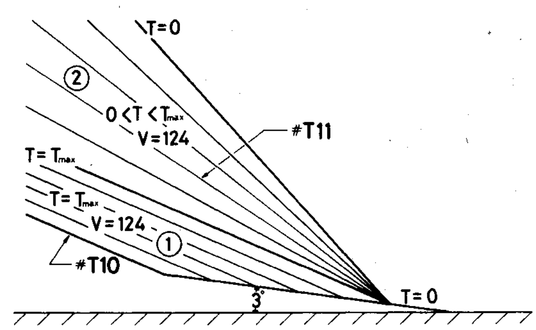


Fig. 2 Flight path and thrust for $V \leq 124$ m/s ($h_0$ free).

An alternative procedure is to restrict the flight velocity below the initial speed, since the aircraft was assumed to be in landing-gear and flaps-down configuration. This constitutes another optimization problem with the state variable constraint:

$$V \leq 124\,\text{m/s} \quad \text{for} \quad t_0 \leq t \leq t_f$$

The solution yields two-segment approach paths of the type illustrated in Fig. 2. Whichever point of region ① an aircraft enters, it must descend with a $\gamma = -3.14$ deg and $T = T_{max}$ until it reaches the final approach path. The thrust is then reduced at the respective point on the final segment. In region ②, the values of $\gamma$ and $T$ depend on the point of entrance. The thrust is constant along each descending path, and is reduced as soon as the path angle is changed to $-3$ deg.

These two considerations give a theoretical basis for the two-segment approach, one of whose aims is to reduce the time required for the landing approach.

## Minimum Noise Landing Approach

Let Eq. (1) be rewritten using an independent variable $x$ and $(\dot{\ }) = \mathrm{d}/\mathrm{d}x$ as

$$\dot{V} = (T - D(V))/mV - g\gamma/V \qquad \dot{h} = \gamma \tag{6}$$

where the boundary conditions and the constraints are the same as in Eqs. (2) and (3).

### Performance Index

In the development of an appropriate performance index it is necessary to estimate the amount of noise inconvenience encountered by people living in the community surrounding an airport, and by the people in the neighborhood of the runway. Jacob[8] expressed the maximum perceived noise (PN dB) level of a STOL aircraft having vectored thrust as

$$\text{PN dB} = \text{PN dB}_0 + 25\log(R_{ref}/R) + 52\log(T/T_{max}) \tag{7}$$

where PN $\text{dB}_0$ is the perceived noise level in dB encountered at a distance of $R_{ref}$ (=152 m) from the aircraft with its maximum thrust, and the second and third terms show the contributions of the distance from the aircraft and of the part thrust. Equation (7) is an estimate based on the results given by Lee et al.[14] Almost the same expression can be derived for the airplane considered in this paper. Assumptions made in the derivation are that the noise field is spherically symmetrical, and that the compressor noise is small compared to the jet noise.

The community surrounding a runway is assumed, according to Ref. 8, to be the area before a point 472 m horizontally distant from the touchdown point, and outside a line 61 m laterally away from the centerline of the runway. When an aircraft is at a horizontal distance $d$ from the touchdown point, $R$ in Eq. (7) is given by

$$R = \begin{cases} h & \text{if } d \geq 472 \\ \sqrt{h^2 + 61^2} & \text{if } d < 472 \end{cases} \tag{8}$$

This expression is difficult to analyze, and it is approximated by $h + 50$.

The effective maximum perceived noise (EPN) level can be calculated by adding the effect of the noise duration to the basic PN dB value.[7]

$$\text{EPN dB} = \text{PN dB} + 20\log(V_{ref}/V) \tag{9}$$

Substituting Eq. (7) into Eq. (9), and rewriting using natural logarithms, a performance index for the "integral maximum effective perceived noise" becomes

$$J_0 = \int_{x_0}^{x_f} [K_1 - K_2 \ln(h+50) + K_3 \ln T - K_4 \ln V]\,\mathrm{d}x \tag{10}$$

where $K_i$'s ($i = 1,\ldots,4$) are positive constants. The index $J_0$ represents the integral value of EPN dB encountered by the people nearest to the airplane. The average value is, dividing $J_0$ by the total horizontal flight distance,

$$\overline{\text{EPN dB}} = \frac{J_0}{x_f - x_0} = \frac{1}{x_f - x_0}\int_{x_0}^{x_f} (\text{EPN dB})\,\mathrm{d}x \tag{11}$$

The index $J_0$ is usually selected as a noise criterion, but it is somewhat deficient. This is due to the use of the noise level measured in dB units and integrated over some distance. The contributions of $h$, $T$, and $V$ to $J_0$ in Eq. (10) are a logarithmic sum, and independent of each other. This implies that the maximum thrust may be used at all altitudes, and that its contribution to $J_0$ is the same, as long as the flight distances with $T_{max}$ are equal. For this reason, the noise level of the optimum solution may have a high peak value.

The optimum solution of $J_0$ is given in the Appendix. As shown in Eq. (A7), the optimum thrust is a bang-bang control, and is the same as that for the minimum time problem. In case of flight along a specified path, noise abatement can not be achieved even if the optimum thrust program is used. A performance index should furnish, therefore, an effective penalty for the use of high thrust at low altitude.

A performance index appropriate for these purposes is not the integral value of EPN dB, but

$$\text{EPN} = 10^{\text{EPN dB}/10}$$

Substituting Eqs. (7) and (9), the performance index to be minimized is expressed by

$$J = \int_{x_0}^{x_f} P(V,h,T)\,\mathrm{d}x \qquad P \triangleq \frac{C_N T^{5.2}}{V^2 (h+50)^{2.5}} \tag{12}$$

where $C_N$ is a constant.

$$C_N = 10^{\text{PN dB}_0/10} \times \frac{R_{ref}^{2.5} V_{ref}^2}{T_{max}^{5.2}} \tag{13}$$

### Optimum Solution of $J$

The Hamiltonian for $J$ in Eq. (12) is

$$H = \lambda_1 [(T - D(V))/mV - g\gamma/V] + \lambda_2 \gamma + P(V,h,T) \tag{14}$$

$H$ is not an explicit function of $x$,

$$\dot{H} = 0 \tag{15}$$

Lagrange multipliers $\lambda_i$'s obey the following equations:

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial V} = -\lambda_1 \frac{\partial}{\partial V}\left(\frac{T - D(V)}{mV} - \frac{g\gamma}{V}\right) + \frac{2P(V,h,T)}{V} \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial h} = 2.5\frac{P(V,h,T)}{h+50} \end{aligned} \tag{16}$$

From the transversality condition,

$$\lambda_{10}, \lambda_{1f}, \lambda_{20}, \text{ and } \lambda_{2f} \text{ are unspecified} \tag{17}$$

$H$ is a linear function of $\gamma$, and a singular solution of $\gamma$ may exist. When the coefficient $\Gamma$ of $\gamma$

$$\Gamma = -\lambda_l g/V + \lambda_2 \tag{18}$$

is zero, the condition of $d\Gamma/dx = 0$ gives

$$T^{5.2} = 5.2\ f_T^{4.2} \frac{2C_l V^2 - D(V)}{1 - 1.25V^2/g(h+50)} \tag{19}$$

where

$$f_T = [-\lambda_l V(h+50)^{2.5}/5.2mC_N]^{1/4.2} \tag{20}$$

Moreover, from $d^2\Gamma/dx^2 = 0$, the equation to be satisfied by $\gamma$ will be complicated. In the following section, Eq. (19) will be used to check the existence of the singular solution when $\Gamma = 0$ occurs. The optimum solution of $T$ will be derived from the condition that $T$ is a minimizing function of $H$.

Summarizing these equations, the optimum controls for the minimum noise problem are

$$\gamma = \begin{cases} \gamma_{min} & \text{if } \Gamma > 0 \\ \gamma_{max} & \text{if } \Gamma < 0 \\ \gamma_{min} \le \gamma \le \gamma_{max} & \text{if } \Gamma = 0 \text{ and Eq. (19)} \end{cases} \tag{21}$$

$$T = \begin{cases} T_{min} & \text{if } f_T < T_{min} \\ f_T & \text{if } T_{min} \le f_T \le T_{max} \\ T_{max} & \text{if } f_T > T_{max} \end{cases} \tag{22}$$

## Results

The same aircraft data as used earlier were used in this problem. The boundary conditions are also the same as in Eq. (5), and the values of the control constraints are

$$300 \le T \le 3420$$

$$-6 \le \gamma \le -3 \quad \text{or} \quad -8 \le \gamma \le 0 \tag{23}$$

The optimum solutions for two kinds of constraints of $\gamma$ will be sought. The value of $T_{min}$, not zero, was given due to the index $J_0$ in Eq. (10). The constants defined in the previous two sections are

$$K_l = 12.7 \qquad K_2 = 10.9 \qquad K_3 = 22.6$$

$$K_4 = 8.69 \qquad C_N = 18.73 \qquad C_T = -12.9$$

Figure 1 illustrates a noise curve for the reference two-segment approach with the reference thrust (Case N0). To examine the adequacy of the index $J$, Case N1, using the optimum thrust of $J_0$, Eq. (A7), is compared with Case N2 using the one of $J$, Eq. (22), where $\gamma$ is specified as the reference angle. These noise curves are shown in Figs. 3 and 4. It is to be noted that the use of the optimum thrust of $J_0$ does not contribute to noise abatement, but is close to the minimum time solution. This is due to the bang-bang control of $T$ as pointed out earlier. It is also apparent that the peak noise is high. Noise abatement in $J_0$ can be achieved only by advancing the switching time, or by choosing $\gamma$ so as to yield a higher altitude.

By using the optimum thrust of $J$, on the other hand, it is seen from Fig. 4 that the noise level in Case N2, compared with N0, exhibits a noise decrease of more than 20% in the average value of EPN, $\overline{EPN}$, and 1 dB in $\overline{EPN}$ dB. The peak value of Case N2 is also low. This is another advantage of the optimum thrust, because the noise abatement effect sensed by the people living in the community near an airport may be expected to be more than the value of the performance index.

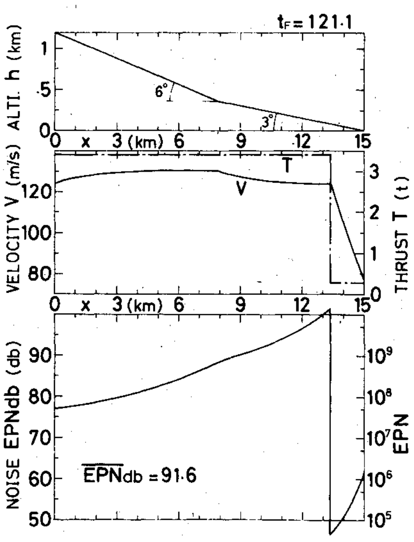


Fig. 3 Case N1 (optimum thrust of $J_0$, the two-segment path).

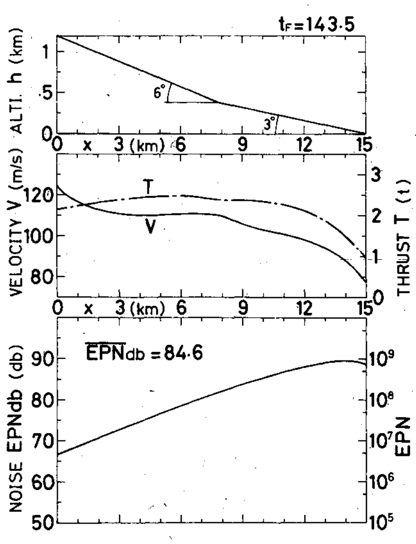


Fig. 4 Case N2 (optimum thrust of $J$, the two-segment path).

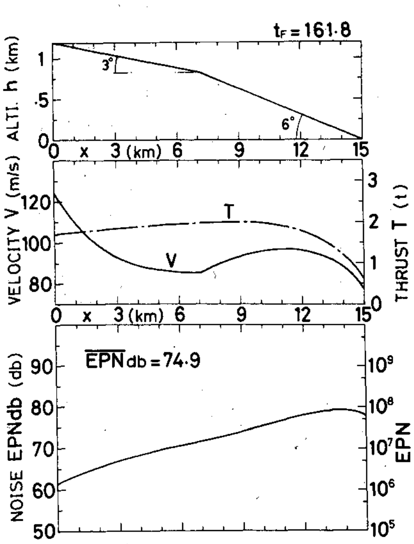


Fig. 5 Case N3 (optimum solution of $J$ for $-6$ deg $\le \gamma \le -3$ deg).

The optimum solutions subject to the constraints given in Eq. (23), Case N3 ($-6$ deg $\leq \gamma \leq -3$ deg), and Case N5 ($-8$ deg $\leq \gamma \leq 0$ deg), were calculated and are illustrated in Figs. 5 and 6. Equation (19) was examined to check the existence of the singular solution of $\gamma$, and was not satisfied. Hence $\gamma$ becomes either $\gamma_{max}$ or $\gamma_{min}$. To minimize the contribution of altitude to $J$ in Case N3, $\gamma$ is initially $\gamma_{max}=-3$ deg, then changed to $\gamma_{min}=-6$ deg to satisfy the boundary conditions. This angle program is opposite to the reference two-segment one. The $\overline{\text{EPN}}$ value of N3 is less than 1/10 that of N0, and its peak value is about 10 dB lower than that of N2. The noise level of the other optimum solution, Case N5 (Fig. 6), exhibits very low values. It is, however, to be noted that the velocity at the midpoint becomes less than the touchdown speed $V_f$, which was selected to be 1.15 $V_{stall}$. An optimization problem with velocity constraints should be considered in this case.

### Optimum Solution with Velocity Constraint

The minimum value of $V$ is assumed to be $V_f$, and the state inequality constraint is denoted by

$$S \triangleq V_{min} - V \leq 0 \tag{24}$$

Considering the velocity curve in Case N5, three possibilities will be examined. When $V$ reaches $V_{min}$ by decelerating from $V_0$ 1) acceleration is made by switching $\gamma$ to $-8$ deg; 2) $\gamma$ switches to $\gamma_s$ so as to continue $V=V_{min}$, where $\gamma_s=[T-D(V_{min})]/mg$ from $\dot{V}=0$; or 3) thrust $T$ is increased so as to continue $V=V_{min}$.

*Case 1.* The result (Case N6) is shown in Fig. 7. The flight path has a short horizontal segment on the ground near the end point. This is not optimum, because $J$ or $\overline{\text{EPN}}$ dB becomes greater than that for Case N3 with narrower constraint of $\gamma$.

*Case 2.* The equations of motion are reduced to $\dot{h}=\gamma$ in this case. The Hamiltonian is

$$H_1=\lambda_2\gamma+P(V_{min}, h, T) \tag{25}$$

The multiplier $\lambda_2$ obeys the same equation as in Eq. (16), and is a monotonous increasing function. Hence, $\gamma$ does not have a singular solution $\gamma_s$ on the boundary of $V$.

*Case 3.* According to Bryson and Ho,[15] the augmented Hamiltonian $H^*$ with the velocity constraint is, introducing another multiplier $\mu$,

$$H^*=H+\mu \mathrm{d}S/\mathrm{d}x \tag{26}$$

where

$$\mu \begin{cases} >0 & S=0 \\ =0 & S<0 \end{cases}$$

The differential equations to be solved are Eq. (6) and

$$\dot{\lambda}_1=-\frac{\partial H^*}{\partial V}=-(\lambda_1-\mu)\frac{\partial}{\partial V}\left(\frac{T-D(V)}{mV}-\frac{g\gamma}{V}\right)+\frac{2P(V,h,T)}{V}$$

$$\dot{\lambda}_2=-\frac{\partial H^*}{\partial h}=2.5\frac{P(V,h,T)}{h+50} \tag{27}$$

The optimum thrust is determined from $\partial H^*/\partial T=0$. When $S=0$, the equation yields

$$T=[-(\lambda_1-\mu)V_{min}(h+50)^{2.5}/5.2mC_N]^{1/4.2} \tag{28}$$

On the other hand, $T$ has to be $D(V_{min})$ on the boundary of $V$, where $\dot{V}=0$ and $\gamma=0$. This relation, and Eq. (28), determines $\mu$ as

$$\mu=\lambda_1+\frac{5.2\ mC_N D(V_{min})^{4.2}}{V_{min}(h+50)^{2.5}} \tag{29}$$

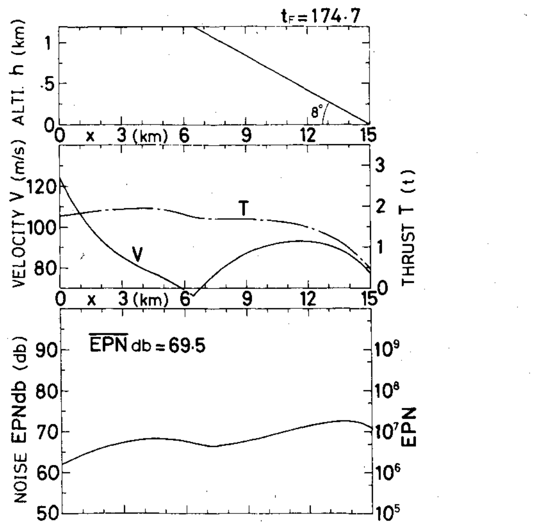


Fig. 6 Case N5 (optimum solution of $J$ without velocity constraint).

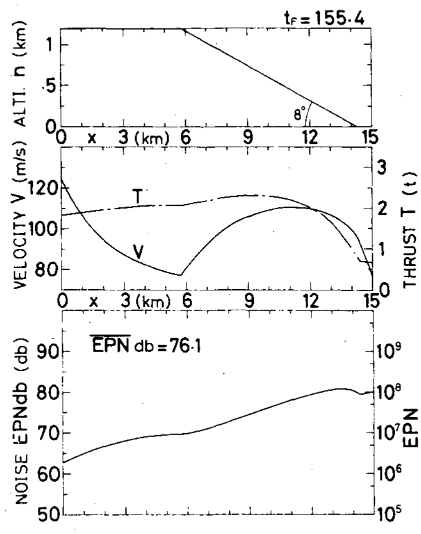


Fig. 7 Case N6 (an example for $V \geq V_{min}$).

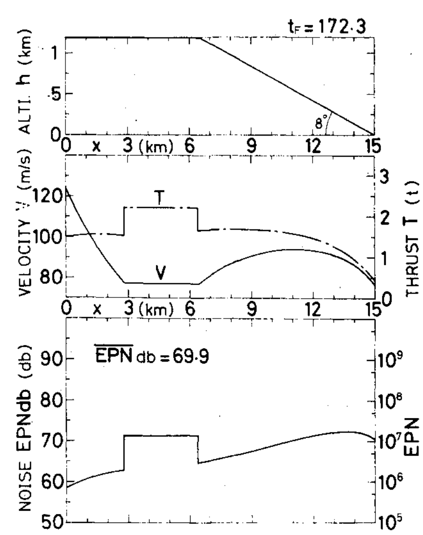


Fig. 8 Case N7 (optimum solution of $J$ for $-8$ deg $\leq \gamma \leq 0$ deg).

Table 1 Solutions of the minimum noise landing problem

| Case no. | $\frac{J}{x_f - x_0} = \overline{\text{EPN}}$ | $\overline{\text{EPN}}$ dB[a] | EPN dB peak | $(t_f - t_0)$, s | Thrust $T$, kg | Fig. no. | $\gamma$, deg | Remarks |
|---|---|---|---|---|---|---|---|---|
| N0 | $3.65\times10^8$ | 85.6 | 97.6 | 153.3 | Reference | 1 | | Ref. sol. for 2-seg. path |
| N1 | $1.44\times10^9$ | 91.6 | 101.4 | 121.1 | $T_{max} \rightarrow T_{min}$ | 3 | $-6 \rightarrow -3$ | Optimum thrust of $J_0$ |
| N2 | $2.85\times10^8$ | 84.6 | 89.6 | 143.5 | Optimum | 4 | specified | Optimum thrust of $J$ |
| N3 | $3.06\times10^7$ | 74.9 | 79.4 | 161.8 | Optimum | 5 | $-6 \le \gamma \le -3$ | Optimum |
| N4 | $1.01\times10^8$ | 80.0 | 88.2 | 126.6 | $T_{max} \rightarrow T_{min}$ | ... | | Optimum of $J_0$ |
| N5 | $8.95\times10^6$ | 69.5 | 73.0 | 174.7 | Optimum | 6 | | Opt. but violates vel. const. |
| N6 | $4.65\times10^7$ | 76.1 | 81.1 | 155.4 | Optimum | 7 | $-8 \le \gamma \le 0$ | Example for $V \ge V_{min}$ |
| N7 | $9.86\times10^6$ | 69.9 | 72.8 | 172.3 | Optimum | 8 | | Optimum |
| N8 | $3.98\times10^8$ | 86.0 | 97.7 | 163.0 | Reference | ... | | Ref. sol. for conv. path |
| N9 | $4.07\times10^8$ | 86.1 | 97.7 | 172.9 | Reference | ... | $-3$ specified | $V_0 = 87.0$ m/s |
| N10 | $3.17\times10^8$ | 85.0 | 89.8 | 153.6 | Optimum | ... | | Optimum thrust of $J$ |

[a] $\overline{\text{EPN}}$ dB = 10 log $\overline{\text{EPN}}$.

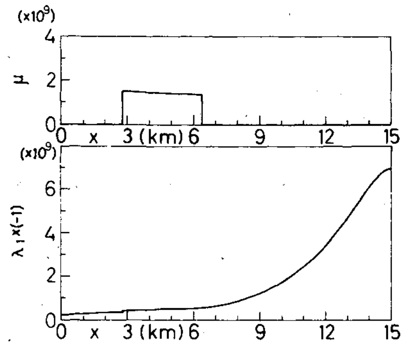

Fig. 9 Multipliers $\lambda_l$ and $\mu$ in Case N7.

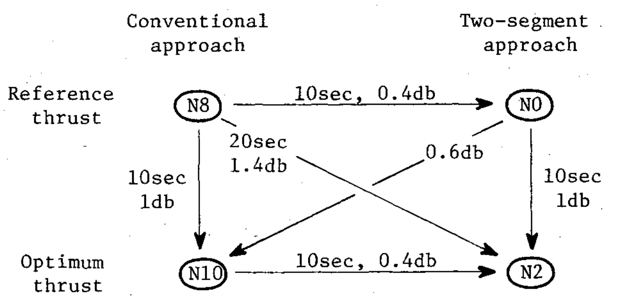

Fig. 10 Comparison of the landing time and $\overline{\text{EPN}}$ dB of the conventional and two-segment approaches.

The result (Case N7) calculated using Eqs. (26-29) is shown in Fig. 8, and is optimum in the case of the velocity constraint. The histories of $\lambda_l$ and $\mu$ are illustrated in Fig. 9, where $\mu = 0$ for $V > V_{min}$, and $\mu > 0$ for $V = V_{min}$. It is to be noted that the multiplier $\lambda_l$ has a little jump at the entry point to the velocity boundary, but there are no discontinuities in the $\lambda$'s at the exit point. (See Ref. 15 for discontinuities of multipliers.)

## Discussion of the Results

The above results are summarized in Table 1 for the purpose of comparison with others. The optimum solution is Case N3 for the constraint $-6$ deg $\le \gamma \le -3$ deg, and Case N7 for $-8$ deg $\le \gamma \le 0$ deg.

To compare these results with the noise levels for a conventional approach path, Cases N8 to N10 were calculated for a specified path angle, $-3$ deg. Cases N8 and N9 used the reference thrust approach, while Case N10 used the optimum one. The initial velocity of N9 was changed to a constant descending speed of 87.0 m/s for the reference thrust. As shown in Case N10, the optimum thrust yields approximately a 20% decrease in $\overline{\text{EPN}}$, or a 1 dB decrease in $\overline{\text{EPN}}$ dB compared with Cases N8 and N9, even if a conventional approach path is used. Case N10 gives a 0.6 dB lower noise level than Case N0 which has a two-segment path. The superiority of the optimum thrust approach is also shown by the fact that the peak level of N10 is lower than that of N0, with almost the same time required for landing. Figure 10 compares the landing time and $\overline{\text{EPN}}$ dB of the conventional and two-segment approaches. The arrows there show the direction of decrease.

Case N4 is the optimum solution of $J_0$ subject to the constraint $-8$ deg $\le \gamma \le 0$ deg. Its $\overline{\text{EPN}}$ dB is lower than that of any other case, N0 to N2 and N8 to N10, with a specified path. Comparing N4 with N1, it is seen that a greater decrease of noise level is gained by advancing the switching time and flying as high as possible. However, N4 is far inferior to N7, the optimum solution of $J$. Its advantages are that the landing time is shorter and the thrust program easier. A disadvantage of the optimum solution, Case N7, is the complexity of the thrust program. Its $\overline{\text{EPN}}$ value, however, is about 1/37 of that of N0, and 1/10 that of N4, that is, a decrease by 10.1 dB in $\overline{\text{EPN}}$ dB is made. Even if it is compared with Case N5 without the velocity constraint, $\overline{\text{EPN}}$ dB of N7 is only a little higher, and its peak level then a little lower.

The noise levels of all of the cases listed in Table 1 are relatively low for the first half of the approach. However, an important portion of the interval for the noise minimum approach is about 5 km before the end point. The optimum solution exhibits a much lower EPN level than the other cases. Another advantage of the optimum solution is that its peak noise level is low. This is one of the important factors for the noise abatement landing.

The characteristic of the optimum thrust is to use its intermediate value over the whole interval. The use of neither maximum nor minimum thrust is advantageous, because the optimum thrust changes continuously unless the velocity reaches its boundary. It is to be noted in Figs. 1 and 3 that the discontinuities on the thrust curves lead to the peak EPN values. For this reason, it can be said that the optimum thrust may be controlled so as to yield as smooth EPN curves as possible. This implies, however, using onboard computer control, because satisfactory manual control would be most difficult to achieve.

It was assumed in this formulation, that the path angle $\gamma$ is $\gamma_{min}$ until touchdown. From safety considerations it is desirable that the aircraft is transferred to the final path of −3 deg at an altitude of approximately 250 m as indicated in the two-segment method discussed by Dunham et al.[16] The descending path with $\gamma_{min}$ to the transfer point will be optimum for this case.

The control of both thrust and path angle is important in the noise problem. Path angle was of minor importance in the minimum time problem. This is obvious from Cases N1 and N4 in Table 1. The times required for landing differ little from the minimum time solutions, but, the minimum noise solutions, N3 and N7, need a much longer time for landing.

## Conclusion

The necessary conditions for determining the optimum thrust and flight path angle programs for minimum time and minimum noise landing approaches were derived. In the minimum time problem, the thrust is a dominant control, and the path angle has a singular solution. Using the optimum program, a decrease of about 20% in the time required for landing is possible. In the minimum noise problem it was found that the integral of EPN (effective maximum perceived noise) level gives a more appropriate performance index than the integral of the EPN dB. The optimum thrust achieves neither its maximum nor its minimum values, but varies continuously so as to achieve a smooth EPN level curve. The average value of the EPN level, $\overline{EPN}$, of the optimum solution is about 1/10 for the flight path angle constraint $-6\ \text{deg} \leq \gamma \leq -3$ deg, and 1/37 for $-8\ \text{deg} \leq \gamma \leq 0$ deg of the reference solution. That is, the noise decreases by 10.7 dB and 15.7 dB in $\overline{EPN}$ dB, respectively. Another advantage of the optimum thrust approach is that the peak noise level is low.

Some additional work which is required on this subject, and which may yield interesting results, includes 1) optimization with a performance index combining time and noise, and 2) optimization of a higher order system which yields a continuous function of the flight path angle. With respect to the latter, a singular perturbation method is now being developed by the author.

## Appendix: Optimum Solution of $J_0$

The Hamiltonian for $J_0$ is, from Eqs. (6) and (10),

$$\begin{aligned} H=&\lambda_1[(T-D(V))/mV-g\gamma/V]+\lambda_2\gamma \\ &+K_1-K_2\ln(h+50)+K_3\ln T-K_4\ln V \end{aligned} \tag{A1}$$

Lagrange multipliers $\lambda_i$'s obey the following equations:

$$\begin{aligned} \dot{\lambda}_1&=-\frac{\partial H}{\partial V}=-\lambda_1\frac{\partial}{\partial V}\left(\frac{T-D(V)}{mV}-\frac{g\gamma}{V}\right)+\frac{K_4}{V} \\ \dot{\lambda}_2&=-\frac{\partial H}{\partial h}=\frac{K_2}{h+50} \end{aligned} \tag{A2}$$

The conditions, Eqs. (15) and (17), are held for this index. $H$ is a linear function of $\gamma$, and a singular solution of $\gamma$ should be examined. When $\Gamma$ in Eq. (18) is zero, the conditions $d\Gamma/dx=0$ and $d\Gamma^2/dx^2=0$ give

$$\frac{2\lambda_2}{m}[D(V)-2C_1V^2]-gK_4+\frac{K_2V^2}{h+50}=0 \tag{A3}$$

$$\gamma=\frac{1}{Q}\left[2\lambda_1D(V)\frac{T-D(V)}{m}-\frac{K_2V^2}{h+50}(T-2C_1V^2)\right] \tag{A4}$$

where $Q$ must be nonpositive from the junction condition, $-\partial/\partial\gamma\,(d^2\Gamma/dx^2)\geq 0$

$$Q=2g\lambda_2D(V)-\frac{mK_2V^2}{h+50}\left[g+\frac{V^2}{2(h+50)}\right]\leq 0 \tag{A5}$$

The path angle $\gamma$ may thus have a singular solution. The optimum thrust, on the other hand, is either $T_{max}$ or $T_{min}$ because of $\partial^2H/\partial T^2=-K_3/T^2<0$.

The optimum control for $J_0$ is summarized as

$$\gamma=\begin{cases} \gamma_{min} & \text{if } \Gamma>0 \\ \gamma_{max} & \text{if } \Gamma<0 \\ \text{Eq. (A4)} & \text{if } \Gamma=0, \text{ and Eqs. (A3) and (A5)} \end{cases} \tag{A6}$$

$$T=\begin{cases} T_{min} & \text{if } C_T<\lambda_2/V \\ T_{max} & \text{if } C_T\geq\lambda_2/V \end{cases} \tag{A7}$$

where

$$C_T\triangleq mK_3\frac{\ln(T_{min}/T_{max})}{T_{max}-T_{min}} \tag{A8}$$

## Acknowledgments

The author is indebted to Dr. P.N. Nikiforuk, Dean of the College of Engineering, and to Dr. M.M. Gupta, Professor of Mechanical Engineering, University of Saskatchewan, for discussing the paper. The work was partly supported by the National Science and Engineering Research Council of Canada under Grants A-5625 and A-1080. The author would also like to express his thanks to Mr. Y. Kusunose and Mr. H. Kato for preparation of the numerical results. Calculations are performed using the digital computer FACOM M-200 in the Computer Center of Nagoya University.

## References

[1] Rutowski, E.S., "Energy Approach to the General Aircraft Performance Problem," *Journal of the Aeronautical Sciences,* Vol. 21, March 1954, pp. 187-195.

[2] Bryson, A.E. and Desai, M.N., "Energy State Approximation in Performance Optimization of Supersonic Aircraft," *Journal of Aircraft,* Vol. 6, Nov.-Dec. 1969, pp. 481-488.

[3] Schultz, R.L. and Zagalsky, N.R., "Aircraft Performance Optimization," *Journal of Aircraft,* Vol. 9, Feb. 1972, pp. 108-114.

[4] Hedrick, J.K. and Bryson, A.E., "Minimum Time Turns for a Supersonic Airplane at Constant Altitude," *Journal of Aircraft,* Vol. 8, March 1971, pp. 182-187.

[5] Erzberger, H. and Lee, H.Q., "Optimum Horizontal Guidance Techniques for Aircraft," *Journal of Aircraft,* Vol. 8, Feb. 1971, pp. 95-101.

[6] Vincent, T.L., Lutze, F., and Ishihara, T., "Application of the Calculus of Variations to Aircraft Performance," NASA CR-499, May 1966.

[7] Erzberger, H. and Lee, H.Q., "Technique for Calculating Optimum Takeoff and Climbout Trajectories for Noise Abatement," NASA TN D-5182, 1969.

[8] Jacob, H.G., "An Engineering Optimization Method with Application to STOL-Aircraft Approach and Landing Trajectories," NASA TN D-6978, Sept. 1972.

[9] Bryson, A.E. and Denham, W.F., "A Steepest Ascent Method for Solving Optimum Programming Problems," *Transactions of ASME, Journal of Applied Mechanics,* Vol. 29, Ser. E., June 1962, pp. 247-257.

[10] Kelley, H.J., "Singular Perturbations for a Mayer Variational Problem," *AIAA Journal,* Vol. 8, June 1970, pp. 1177-1178.

[11] Calise, A.J., "A Singular Perturbation Analysis of Optimal Aerodynamic and Thrust Magnitude Control," *IEEE Transactions on Automatic Control,* Vol. AC-24, Oct. 1977, pp. 720-730.

[12] Ohta, H., "Preliminary Analysis of Minimum Time and Minimum Noise Landing Approach Trajectories," AIAA Paper 80-1598, Aug. 1980.

[13] Kelley, H.J., Kopp, R.E., and Moyer, H.G., "Singular Extremals," *Topics in Optimization,* edited by G. Leitman, Academic Press, New York, 1967, pp. 63-101.

[14] Lee, R., Farrell, J., Henry, G., and Lowe, A., "Procedures for Estimating the Effects of Design and Operational Characteristics of Jet Aircraft on Ground Noise," NASA CR-1053, June 1968.

[15] Bryson, A.E. and Ho, Y.C., *Applied Optimal Control,* Blaisdell Publishing Co., Waltham, Mass., 1969, pp. 90-127.

[16] Dunham, R.E., Hatfield, J.J., Elkins, H.C., and Sawyer, R.H., "Flight Evaluation of a Display for Steep Approach," NASA SP-270, May 1971, pp. 177-187.